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A complex deformation of the classical gravitational many-body problem that features many completely periodic motions

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Abstract
A complex deformation of the Newtonian equations of motion of the classical gravitational many-body problem is introduced, namely a many-body problem that features a parameter Ω and that reduces, when this parameter vanishes, to the standard equations of motion of Newtonian gravitation for an arbitrary number of pointlike bodies with arbitrary masses; and it is shown that when this parameter is instead *positive*, $\Omega > 0$, there is an open set of (complex) initial data such that *all* the (complex) motions originating from it are *completely periodic* with period $T = 2\pi/\Omega$, and that the (infinite) measure of this set is a finite (nonvanishing) fraction of the measure of the entire set of initial data.

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1. Introduction and main result

Recently a 'trick'—amounting essentially to a convenient change of variables—has been introduced [1], and it has been exploited to demonstrate the periodic character of solutions of known (integrable and nonintegrable) many-body problems [2–10], as well as to introduce a (complex) deformation of evolution equations (integrable and nonintegrable ODEs and PDEs) that causes the deformed equations to possess many *completely periodic* solutions [11–14]. The purpose and scope of this paper is to pursue the applicability of this approach to the classical ('doubly Newtonian') equations of motion of the many-body gravitational problem, by presenting a result that is backed here by a complete proof but has already been (almost) stated in [8] (see in particular exercise 5.6.5-20 in this book; but beware of a misprint in the left-hand side of equation (5.6.5-41a): the term $2/p$ should instead read $p/2$).

Consider the many-body problem characterized by the following equations of motion:

$$\ddot{\vec{r}}_n + \mathrm{i}\Omega\dot{\vec{r}}_n + 2\Omega^2\vec{r}_n = \sum_{m=1, m\neq n}^{N} M_m(\vec{r}_m - \vec{r}_n)r_{mn}^{-3}, \qquad r_{mn} = [(\vec{r}_m - \vec{r}_n)\cdot(\vec{r}_m - \vec{r}_n)]^{1/2}. \quad (1.1)$$

Here (and throughout this paper) superposed arrows identify three-vectors, say $\vec{r} \equiv (x, y, z)$, the subscript n runs over the integers from 1 to N (with N an *arbitrary positive integer*, $N \geqslant 2$) so that the three-vectors $\vec{r}_n \equiv \vec{r}_n(t)$ identify the positions of the N moving pointlike bodies, superposed dots denote differentiations with respect to the time variable t (which is hereafter assumed to be *real*), say $\dot{\vec{r}}_n \equiv \dot{\vec{r}}_n(t) \equiv \mathrm{d}\vec{r}_n(t)/\mathrm{d}t$, the symbol i identifies the imaginary unit (namely the square root of the negative unit, $\mathrm{i}^2 = -1$), Ω is a *real* constant (which, without loss of generality, is hereafter assumed to be *positive*, $\Omega > 0$), the quantities M_n are the masses of the N moving bodies (which could even be *complex* numbers, without spoiling the validity of the following results, see (2.18)) and the dots sandwiched between two three-vectors denote the standard scalar product, so that, say, $\vec{r}\cdot\vec{r} = r^2 = x^2 + y^2 + z^2$.

Clearly for $\Omega = 0$ these equations of motion, (1.1), characterize the standard gravitational N-body problem, with units chosen so that the gravitational constant equals unity. We consider instead the case with *positive* Ω, $\Omega > 0$, when, due to the second term in the left-hand side of (1.1), the motion takes place in the *complex*, hence we assume hereafter that the three components of the vector $\vec{r}_n$ are *complex* numbers. This fact, and the presence of the second and third additional terms in the left-hand side of (1.1), entail that the many-body problem (1.1) we consider differs substantially from the standard gravitational many-body problem, although clearly it can be seen as a complex deformation of it due to the presence of the parameter Ω. Indeed, as the result we are about to state indicates, one might (or perhaps should) rather view the many-body problem (1.1) as describing N three-dimensional *complex* oscillators interacting nonlinearly via two-body forces 'of gravitational type', or equivalently $2N$ three-dimensional *real* oscillators interacting likewise, as implied by the following reformulation of (1.1), which is clearly obtained by introducing the real and imaginary parts of the complex three-vectors $\vec{r}_n = \vec{u}_n + \mathrm{i}\vec{v}_n$:

$$\ddot{\vec{u}}_n - \Omega\dot{\vec{v}}_n + 2\Omega^2\vec{u}_n = \mathrm{Re}\left\{\sum_{m=1, m\neq n}^{N} M_m[\vec{u}_{mn} + \mathrm{i}\vec{v}_{mn}][u_{nm}^2 - v_{mn}^2 + 2\mathrm{i}\vec{u}_{mn}\cdot\vec{v}_{mn}]^{-3/2}\right\}, \quad (1.2a)$$

$$\ddot{\vec{v}}_n + \Omega\dot{\vec{u}}_n + 2\Omega^2\vec{v}_n = \mathrm{Im}\left\{\sum_{m=1, m\neq n}^{N} M_m[\vec{u}_{mn} + \mathrm{i}\vec{v}_{mn}][u_{nm}^2 - v_{mn}^2 + 2\mathrm{i}\vec{u}_{mn}\cdot\vec{v}_{mn}]^{-3/2}\right\}, \quad (1.2b)$$

where of course $\vec{u}_n \equiv \vec{u}_n(t)$ and $\vec{v}_n \equiv \vec{v}_n(t)$ are *real* three-vectors and we use the short-hand notation $\vec{u}_{mn} \equiv \vec{u}_m - \vec{u}_n$, $\vec{v}_{mn} \equiv \vec{v}_m - \vec{v}_n$.

Be this as it may, we are now ready to state our result.

Proposition. *If the initial data for the problem (1.1) satisfy appropriate inequalities, the corresponding trajectories are completely periodic with period*

$$T = 2\pi/\Omega, \quad (1.3a)$$

$$\vec{r}_n(t+T) = \vec{r}_n(t). \quad (1.3b)$$

The 'appropriate inequalities' mentioned in this proposition are explicitly exhibited below and in more detail in the following section 2, where a complete proof of this proposition is provided. Now we outline the basic idea that underlies this proof, by recalling the 'trick' mentioned above and by thereby making the validity of this proposition rather evident.

We set

$$\vec{r}_n(t) = \exp(-2\mathrm{i}\Omega t)\vec{\rho}_n(\tau), \tag{1.4a}$$

$$\tau = [\exp(3\mathrm{i}\Omega t) - 1]/(3\mathrm{i}\Omega). \tag{1.4b}$$

It is then plain that the equations of motion (1.1) take the new form

$$\vec{\rho}_n'' = \sum_{m=1, m\neq n}^{N} M_m(\vec{\rho}_m - \vec{\rho}_n)\rho_{mn}^{-3}, \qquad \rho_{mn} = [(\vec{\rho}_m - \vec{\rho}_n)\cdot(\vec{\rho}_m - \vec{\rho}_n)]^{1/2}, \tag{1.5}$$

that only differs from (1.1) because of the absence of the 'Ω-terms' in the left-hand sides of (1.5), and of course because the superposed dots are now replaced by appended primes, which obviously signify differentiations with respect to the new independent variable τ, $\vec{\rho}_n'(\tau) \equiv \mathrm{d}\vec{\rho}_n(\tau)/\mathrm{d}\tau$.

The following relations among the initial data for (1.1) and (1.5) are moreover evident:

$$\vec{r}_n(0) = \vec{\rho}_n(0), \tag{1.6a}$$

$$\dot{\vec{r}}_n(0) = \vec{\rho}_n'(0) - 2\mathrm{i}\Omega\vec{\rho}_n(0). \tag{1.6b}$$

As we mentioned above, we always consider the evolution of the system (1.1) when the 'physical' time variable t is *real*. It is plain from (1.4b) that the corresponding evolution of the *complex* timelike variable τ, as t evolves starting from $t = 0$ onward, is to travel round and round over the circle $\tilde{C}$ the diameter of which lies on the upper imaginary axis in the complex τ-plane, from $\tau = 0$ (corresponding to $t = 0 \bmod(T/3)$) to $\tau = 2\mathrm{i}/(3\Omega)$ (corresponding to $t = T/6 \bmod(T/3)$). Hence whenever a solution $\vec{\rho}_n(\tau)$ of (1.5) is *holomorphic* (or, for that matter, just *meromorphic*), as a function of the *complex* variable τ, in the (closed) circular disc C enclosed by the circle $\tilde{C}$, the corresponding solution $\vec{r}_n(t)$ of (1.1) is *completely periodic* in t with period T (see (1.4) and (1.3)).

On the other hand the general theorem that guarantees existence, uniqueness and analyticity of the solutions of (systems of) analytic ODEs in the neighbourhood of their initial data entails that there exists a circular disc D centred, in the complex τ-plane, at $\tau = 0$, inside which the solution $\vec{\rho}_n(\tau)$ of (1.5) is *holomorphic*. The corresponding solution $\vec{r}_n(t)$ of (1.1) is therefore guaranteed to be *completely periodic* in t with period T provided the radius τ_c of the disc D is larger than the diameter of the disc C,

$$\tau_c > 2/(3\Omega), \tag{1.7}$$

so that the disc D includes the disc C, implying that the solution $\vec{\rho}_n(\tau)$ of (1.5) is *holomorphic* in the closed disc C.

Clearly the value of the radius τ_c depends, for a system of ODEs of type (1.5), essentially on two aspects of the *initial* data: the magnitude of the initial rates of change $\vec{\rho}_n'(0)$ of the dependent variables, and the overall initial magnitude of the 'forces' that constitute the right-hand sides of the ODEs (1.5). Hence for the system (1.5) the following *positive* parameters characterizing the initial data play a key role: a parameter V that provides an upper bound for the first type of data,

$$V = \max_{n=1,\ldots,N;\, j=1,2,3} |\rho_{n,j}'(0)|, \tag{1.8}$$

and three parameters r, R, Q (the role of which is detailed below) that bound the initial 'interparticle distances',

$$r = \min_{n,m=1,\ldots,N, n\neq m;\, j=1,2,3} |\mathrm{Re}\,[\rho_{n,j}(0) - \rho_{m,j}(0)]|, \tag{1.9a}$$

$$R = \max_{n,m=1,\ldots,N, n\neq m;\, j=1,2,3} |\mathrm{Re}\,[\rho_{n,j}(0) - \rho_{m,j}(0)]|, \tag{1.9b}$$

$$Q = \max_{n,m=1,\ldots,N, n\neq m;\, j=1,2,3} |\mathrm{Im}\,[\rho_{n,j}(0) - \rho_{m,j}(0)]|, \tag{1.9c}$$

and are therefore instrumental in obtaining an upper bound for the 'forces'. Note that in these two definitions, (1.8) and (1.9), and always below, the 'modulus' symbol, $|z|$, denotes the modulus of the *complex* number z (not the modulus of a vector), while of course the subscript j in (1.8) and (1.9), and always below, identifies the three components of the relevant three-vector. (Of course the definitions (1.9*a*), (1.9*b*) entail the inequality $R \geqslant (N-1)r$, which however will play no role in the following; also note that the requirement that r be *positive*, $r > 0$, is not automatically entailed by its definition, but rather by the following developments, see (1.11*b*).)

It stands to reason that the radius τ_c of the disc D associated with a solution $\vec{\rho}_n(\tau)$ of (1.5) can be made *arbitrarily* large—and in particular consistent with the inequality (1.7)—by requiring that the initial data that determine this solution be characterized by a not too large value of V (see (1.8)) and by sufficiently large (and appropriately ranked) values of the three quantities r, R, Q (see below). Hence, for all such initial data, the fact that the solution $\vec{\rho}_n(\tau)$ of (1.5) is *holomorphic* in D entails that it is also *holomorphic* in C (which is then included in D), and this entails that the corresponding solution $\vec{r}_n(t)$ of (1.1) (see (1.4)) is *completely periodic* with period T (see (1.3)). Hence *all* the solutions of (1.1) that emerge from initial data $\vec{r}_n(0), \dot{\vec{r}}_n(0)$ that correspond, via (1.6), to such initial data $\vec{\rho}_n(0), \vec{\rho}\,'_n(0)$ (that guarantee the validity of (1.7)) yield *completely periodic* solutions of (1.1), and it will be immediately clear that these data—that are characterized by the required validity of certain *inequalities* (see below)—are a set the measure of which constitutes a finite (nonvanishing) fraction of the entire universe of initial data for problem (1.5) as well as (1.1).

In fact the following conditions on the parameters V, r, R, Q characterizing the initial data, see (1.9), are *sufficient* to guarantee validity of the proposition stated above. Set

$$V = \bar{V}\varepsilon^{-q}, \tag{1.10a}$$

$$r = \bar{r}\varepsilon^{-p}, \qquad R = \bar{R}\varepsilon^{-p}, \qquad Q = \bar{Q}\varepsilon^{-p}, \tag{1.10b}$$

where ε is a small parameter and $\bar{V}, \bar{r}, \bar{R}, \bar{Q}$ are independent of ε, so that the two real exponents p, q characterize the magnitude of the parameters V, r, R, Q. Then, as proven in the following section 2, provided

$$\bar{r} > \bar{Q} \tag{1.11a}$$

namely

$$r > Q, \tag{1.11b}$$

the two simple conditions

$$p > 1/2, \qquad q < p, \tag{1.12}$$

are *sufficient* to guarantee that, for any arbitrarily assigned (up to (1.11*a*)) values of the positive constants $\bar{V}, \bar{r}, \bar{R}, \bar{Q}$ and Ω, there is a finite (nonvanishing) value ε_c (depending of course on these parameters, and on the parameter M, see (2.18)) such that, for $0 < \varepsilon < \varepsilon_c$, the inequalities (1.9) with (1.10) guarantee validity of the inequality (1.7), and hence of the assertion (see (1.3)) made in the above proposition.

Note that, when q, in addition to p, is chosen to be positive (as permitted by (1.12)), the initial data consistent with these requirements can be arbitrarily large, hence the set of data $\vec{\rho}_n(0), \vec{\rho}\,'_n(0)$ satisfying these restrictions (see (1.10) and (1.9)) has a measure that constitutes a finite (nonvanishing) fraction of the entire universe of such initial data; and clearly the same assertion holds for the set of initial data $\vec{r}_n(0), \dot{\vec{r}}_n(0)$, related via (1.6) to $\vec{\rho}_n(0), \vec{\rho}\,'_n(0)$, which thereby guarantee validity of the assertion (see (1.3)) made in the above proposition. Here and above the measure we refer to is of course any reasonable one associated with the (complex) initial data $\vec{\rho}_n(0), \vec{\rho}\,'_n(0)$ or, equivalently via (1.6), $\vec{r}_n(0), \dot{\vec{r}}_n(0)$.

This concludes this introductory section, in which we have stated, and made quite plausible or perhaps even obvious, the main result of this paper, as formulated in the above proposition. A detailed proof is provided in the following section 2, and some final remarks are in section 3.

2. Proof

Our treatment here is patterned after section 3 of [9] (to the extent allowed by the three-dimensional character of the problem treated herein, which entails significant complications, see below). We rely on the standard theorem guaranteeing the existence, uniqueness and analyticity of the solution of the initial-value problem for systems of analytic ODEs (see for instance [15]). It is therefore convenient to reformulate the equations of motion (1.5) so that they conform to the notation of [15]. Hence we set

$$w_{n,j}(\tau) = \rho_{n,j}(\tau) - \rho_{n,j}(0), \qquad n = 1, \ldots, N, \ j = 1, 2, 3, \quad (2.1a)$$

$$w_{N+n,j}(\tau) = \vartheta[\rho'_{n,j}(\tau) - \rho'_{n,j}(0)], \qquad n = 1, \ldots, N, \ j = 1, 2, 3, \quad (2.1b)$$

where ϑ is a *positive* rescaling constant, $\vartheta > 0$, the value of which will be chosen at our convenience later, while we trust the rest of the notation to be self-evident (the subscript j identifies of course the three components of a three-vector). Thereby our system (1.5) of N second-order three-vector ODEs (equivalent to a system of $3N$ second-order scalar ODEs) becomes the following (standard) system of $6N$ first-order ODEs:

$$w'_{l,j} = f_{l,j}(\underline{w}), \qquad l = 1, \ldots, 2N, \ j = 1, 2, 3 \quad (2.2)$$

with

$$f_{n,j}(\underline{w}) = \rho'_{n,j}(0) + w_{N+n,j}/\vartheta, \qquad n = 1, \ldots, N, \ j = 1, 2, 3, \quad (2.3a)$$

$$f_{N+n,j}(\underline{w}) = \vartheta \sum_{m=1, m\neq n}^{N} M_m[\rho_{m,j}(0) - \rho_{n,j}(0) + w_{m,j} - w_{n,j}]S_{nm}^{-3/2},$$
$$n = 1, \ldots, N, \ j = 1, 2, 3, \quad (2.3b)$$

where

$$S_{nm} \equiv S_{nm}(\underline{w}) = \sum_{k=1}^{3} [\rho_{m,k}(0) - \rho_{n,k}(0) + w_{m,k} - w_{n,k}]^2. \quad (2.4)$$

In (2.2), (2.3), (2.4) and below we use the short-hand notation $\underline{w}$ to denote the $(6N)$-vector the components of which are the $6N$ quantities $w_{l,j}$, see (2.1).

Note that the definition (2.1) also entails that the new dependent variables all vanish initially, $w_{n,j}(0) = w_{N+n,j}(0) = 0$ (consistent with the notation of [15]).

The standard result [15] then provides the following lower bound for the radius τ_c of the circular disc D, centred at the origin $\tau = 0$ in the complex τ-plane, within which the solutions $w_{n,j}(\tau)$ of (2.2) with (2.3), and hence also (see (2.1a)) the solutions $\rho_{n,j}(\tau)$ of (1.5), are *holomorphic*:

$$\tau_c > W/[(6N + 1)F] \quad (2.5)$$

(this formula coincides, up to trivial notational changes, with the last equation of section 12.21 of [15], with the assignments $m = 6N$ and $a = \infty$, the first of which is justified by the fact that (2.2) features $6N$ ODEs and the second of which is justified by the *autonomous* character of our equations of motion). The two *positive* quantities W and F in this equation, (2.5), are defined as follows. The quantity W is required to guarantee that the $6N$ quantities $f_{l,j}$ (see (2.3)) are all *holomorphic* when there hold the $6N$ inequalities

$$|w_{l,j}| \leqslant W, \qquad l = 1, \ldots, 2N, \ j = 1, 2, 3. \quad (2.6)$$

The quantity $F \equiv F(W)$ is the upper bound of the moduli of the (complex) quantities $f_{l,j}$ (see (2.2) and (2.3) with (2.4)) when the quantities $w_{l,j}$ satisfy the restriction (2.6),

$$F(W) = \max_{l=1,\ldots,2N, j=1,2,3; |w_{l,j}|<W} |f_{l,j}(\underline{w})|; \tag{2.7}$$

but of course the inequality (2.5) holds *a fortiori* if we overestimate $F(W)$ (as we shall do, see below).

Our first task now is to ascertain how W must be limited in order to be consistent with its definition (see (2.6)). Clearly the only source of singularities of the $6N$ quantities $f_{l,j}$, see (2.3), is the vanishing of the quantities S_{nm} (see (2.4)). Hence a *sufficient* condition to exclude the occurrence of such singularities is the requirement that the real part of S_{nm},

$$\mathrm{Re}\,[S_{nm}] = \sum_{k=1}^{3}\{(\mathrm{Re}\,[\rho_{m,k} - \rho_{n,k} + w_m - w_n])^2 - (\mathrm{Im}\,[\rho_{m,k} - \rho_{n,k} + w_m - w_n])^2\}, \tag{2.8a}$$

$$\begin{aligned}\mathrm{Re}\,[S_{nm}] = \sum_{k=1}^{3}\{&(\mathrm{Re}\,[\rho_{m,k} - \rho_{n,k}])^2 - (\mathrm{Im}\,[\rho_{m,k} - \rho_{n,k}])^2 \\ &+ 2\mathrm{Re}\,[w_m - w_n]\mathrm{Re}\,[\rho_{m,k} - \rho_{n,k}] - 2\mathrm{Im}\,[w_m - w_n]\mathrm{Im}\,[\rho_{m,k} - \rho_{n,k}] \\ &+ (\mathrm{Re}\,[w_m - w_n])^2 - (\mathrm{Im}\,[w_m - w_n])^2\},\end{aligned} \tag{2.8b}$$

be *positive*,

$$\mathrm{Re}\,[S_{nm}] > 0. \tag{2.9}$$

(Here and below we take advantage of two obvious properties of a complex number: its modulus can never be smaller than its real part, nor can it exceed the sum of its real and imaginary parts.)

It is now plain, from (2.8*b*) and the definitions (1.9) and (2.6), that

$$\mathrm{Re}\,[S_{nm}] \geqslant S, \tag{2.10}$$

with

$$S = 3\{r^2 - Q^2 - 4W(R + Q) - 4W^2\}, \tag{2.11a}$$

$$S = 12(W_+ - W)(W - W_-), \tag{2.11b}$$

where of course $W_\pm$ are the two roots of the second-degree polynomial in W in the right-hand side of (2.11*a*),

$$W_\pm = [(R + Q)/2]\{\pm [1 + (r^2 - Q^2)/(R + Q)^2]^{1/2} - 1\}. \tag{2.12}$$

Hereafter we assume that (the initial data are such, see (1.9), that) there holds the inequality (1.11*b*), which clearly entails that the quantities $W_\pm$ are real and satisfy the inequalities

$$W_- < 0 < W_+. \tag{2.13}$$

Hence any choice of the *positive* parameter W is hereafter allowed, provided it falls in the interval

$$0 < W < W_+, \tag{2.14}$$

a restriction that clearly guarantees (see (2.11*b*)) that S is *positive*, $S > 0$, hence, via (2.10), the validity of (2.9).

Our next task is to provide, for W in this interval (2.14), an *upper bound* to $F(W)$, and it is clear from (2.7) with (2.3) and (2.8, 10, 11), together with the definitions (1.9), that this is provided by the inequality

$$F(W) \leqslant F_{\max}(W) \tag{2.15}$$

with

$$F_{\max}(W) = \max\,[V + u,\, G(W)/u], \tag{2.16}$$

where

$$G(W) = 12^{-3/2}(N-1)MW(R+Q+2W)[(W_+ - W)(W - W_-)]^{-3/2}, \tag{2.17}$$

$$M = \max_{n=1,\dots,N} |M_n|. \tag{2.18}$$

Here (merely for notational convenience) we have replaced, in the right-hand side of (2.16), the arbitrary *positive* constant ϑ with the arbitrary *positive* constant $u = W/\vartheta$.

The maximum in the right-hand side of (2.16) must be taken over the two arguments of the Max function, hence at this point it is natural to choose the *positive* quantity u so that the values of these two arguments coincide,

$$u = 2G(W)\{V + [V^2 + 4G(W)]^{1/2}\}^{-1}, \tag{2.19}$$

entailing

$$F_{\max}(W) = \{V^2 + V[V^2 + 4G(W)]^{1/2} + 2G(W)\}\{V + [V^2 + 4G(W)]^{1/2}\}^{-1}. \tag{2.20}$$

We are now ready to obtain, via (2.5), a *lower bound* for the radius τ_c of the disc D:

$$\tau_c \geqslant \tau_{\mathrm{lo}}, \tag{2.21}$$

$$\tau_{\mathrm{lo}} = (6N+1)^{-1}W\{V + [V^2 + 4G(W)]^{1/2}\}\{V^2 + V[V^2 + 4G(W)]^{1/2} + 2G(W)\}^{-1} \tag{2.22}$$

where $G(W)$ is of course defined by (2.17) and we are still free to choose W in the interval (2.14).

To show that τ_{lo} can be made arbitrarily large provided the parameters V, r, R, Q characterizing the initial data (see (1.9)) are chosen appropriately we now set

$$W = W_+(1-\varepsilon), \tag{2.23}$$

with W_+ defined of course by (2.12) and ε a *positive* parameter which is hereafter supposed to be small and which will also determine the size of the parameters V, r, R, Q according to the assignments (1.10), with p, q two real parameters, the values of which are discussed below. Then clearly, as $\varepsilon \to 0^+$ (and keeping hereafter, for simplicity, only the leading terms in this limit)

$$W = W_+ = \bar{W}\varepsilon^{-p}, \tag{2.24a}$$

$$\bar{W} = [(\bar{R} + \bar{Q})/2]\{[1 + (\bar{r}^2 - \bar{Q}^2)/(\bar{R} + \bar{Q})^2]^{1/2} - 1\}, \tag{2.24b}$$

and from (2.17) and (2.24)

$$G = \bar{G}\varepsilon^{(2p-3)/2}, \tag{2.25a}$$

$$\bar{G} = 12^{-3/2}(N-1)M\bar{W}^{-1/2}(\bar{R} + \bar{Q})^{-1/2}[1 + (\bar{r}^2 - \bar{Q}^2)/(\bar{R} + \bar{Q})^2]^{-1/4}, \tag{2.25b}$$

hence, from (2.22),

$$\begin{aligned}\tau_{\mathrm{lo}} &= (6N+1)^{-1}\bar{W}\varepsilon^{-p}\{\bar{V}\varepsilon^{-q} + [\bar{V}^2\varepsilon^{-2q} + 4\bar{G}\varepsilon^{(2p-3)/2}]^{1/2}\}\\ &\quad \{\bar{V}^2\varepsilon^{-2q} + \bar{V}\varepsilon^{-q}[\bar{V}^2\varepsilon^{-2q} + 4\bar{G}\varepsilon^{(2p-3)/2}]^{1/2} + 2\bar{G}\varepsilon^{(2p-3)/2}\}^{-1}.\end{aligned} \tag{2.26}$$

Let us now consider first the assignment

$$q < (3-2p)/4. \tag{2.27a}$$

Then

$$\tau_{\mathrm{lo}} = \bar{\tau}_{\mathrm{lo}}\varepsilon^{-(3/2)(p-1/2)}, \tag{2.27b}$$

$$\bar{\tau}_{\mathrm{lo}} = (6N+1)^{-1}\bar{W}\bar{G}^{-1/2}, \tag{2.27c}$$

and we may therefore conclude that, as $\varepsilon \to 0^+$, τ_{lo} diverges provided

$$p > 1/2. \tag{2.28}$$

Clearly the same conclusion is obtained, provided this inequality (2.28) holds, if

$$q = (3 - 2p)/4, \tag{2.29a}$$

except that in this case

$$\tau_{\mathrm{lo}} = \tilde{\tau}_{\mathrm{lo}} \varepsilon^{-(3/2)(p-1/2)}, \tag{2.29b}$$

$$\tilde{\tau}_{\mathrm{lo}} = (6N+1)^{-1} \bar{W} \{\bar{V} + [\bar{V}^2 + 4\bar{G}]^{1/2}\} \{\bar{V}^2 + \bar{V}[\bar{V}^2 + 4\bar{G}]^{1/2} + 2\bar{G}\}^{-1}. \tag{2.29c}$$

The same conclusion is also obtained if

$$q > (3 - 2p)/4, \tag{2.30a}$$

provided

$$q < p, \tag{2.30b}$$

since in this case, as $\varepsilon \to 0^+$,

$$\tau_{\mathrm{lo}} = \hat{\tau}_{\mathrm{lo}} \varepsilon^{q-p}, \tag{2.30c}$$

$$\hat{\tau}_{\mathrm{lo}} = (6N+1)^{-1} \bar{W}/\bar{V}. \tag{2.30d}$$

However, (2.30*a*) with (2.30*b*) clearly implies again (2.28), so in order for the main conclusion to ensue (namely, divergence of τ_{lo} as $\varepsilon \to 0^+$) it is *sufficient* that (2.28) and (2.30*b*) hold, these being the less stringent conditions one obtains with this rather simple treatment.

This concludes our proof of the proposition of section 1.

3. Outlook

As we emphasized in section 1, the finding reported in this paper provides no direct information on the classical gravitational many-body problem, and its validity is fairly obvious—as is indeed the case for *all* mathematically correct results *after* they have been thoroughly understood. Yet, the system under consideration is a deformation of the *realistic* classical gravitational many-body problem—indeed for just this reason we chose as setting for our presentation the ordinary three-dimensional space in which we spend our (classical—who knows nowadays about their quantum counterparts?) lives, while it would have been easy to work in a space with an arbitrary number of dimensions, and also to treat more general interactions than the gravitational one as is for instance done in exercise 5.6.5-20 of [8]. Moreover the result we proved, obvious as it is once its origin has been well understood, does feature certain aspects that might instead cause *a priori* disbelief about its validity: inasmuch as any model that features such abundance of *completely periodic* trajectories, emerging from an *open* set of initial data that constitute a finite (nonvanishing) fraction of the *entire universe* of initial data, might well be conjectured to be *integrable* (rather than to feature just a region of phase space in which it behaves as a—particularly simple—*integrable* system), and such a property of *integrability* seems hardly consistent with a system that, as a deformation of the classical many-body gravitational problem, might be expected to be even *more general* than the many-body problem of Newtonian gravitation, itself a classic example of a *nonintegrable* system.

An interesting problem we plan to investigate further is the possible existence, for the system (1.1), of other *completely periodic* motions, having a period that is an *integer* multiple of T (see (1.3*a*)).

Another, more interesting but possibly untreatable, open problem to which we plan to devote additional study is whether the approach employed in this paper could be used to evince information on the behaviour of the gravitational many-body problem itself, rather than only on its deformed and complexified version (1.1) with $\Omega > 0$.

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